

FUNDAMENTAL INTERACTIONS

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(December 1, 2000, revised Dec. 14, 2000)

Abstract

A generalized Noether's theorem and the operational determination of a physical geometry in quantum physics are used to motivate a quantum geometry consisting of relations between quantum states that are defined by a universal group. Making these relations dynamical implies the non local effect of the fundamental interactions on the wave function, as in the Aharonov-Bohm effect and its generalizations to non Abelian gauge fields and gravity. The usual space-time geometry is obtained as the classical limit of this quantum geometry using the quantum state space metric.

Paper presented at the XXIII International Colloquium on Group Theoretical Methods in Physics, Dubna, Russia, 30 July - 05 August, 2000, [quant-ph/0012011](https://arxiv.org/abs/quant-ph/0012011).

I. INTRODUCTION

The space-time geometry that is commonly used today arose from classical physics. An interesting question is what geometry is appropriate for quantum physics. It was suggested that the universal symmetry group elements which act on all Hilbert spaces may be appropriate for constructing a physical geometry for quantum theory [1]. I also proposed the systematic study of all the fundamental interactions *operationally* from their effects on quantum interference [2]. The purpose of this paper is to attempt to bring together these two approaches. The modular variables introduced by Aharonov, Pendleton and Petersen [3] will play a useful role in this.

In section 2, I shall review Noether's theorem and its converse in a generalized form in which the conserved quantities are elements of a group and not the generators of this group as usually stated. This will suggest a quantum geometry by relations defined by the universal group elements, which constitute the symmetry group of physics, that act on all Hilbert spaces, as discussed in section 3. The classical limit of this geometry will be obtained in section 4 from the quantum state metric in Hilbert spaces and the universality of the action of the translational group in every Hilbert space. This gives the usual Euclidean metrics in physical space and time. In section 5, the non locality of fundamental interactions in quantum physics implied by this approach, as shown physically by the Aharonov-Bohm (AB) effect [4] and its generalizations [9,2], will be studied. The study of the gravitational AB effect around a cosmic string in particular suggests that the use of universal group elements as quantum distances may be appropriate.

II. SOME REFLECTIONS ON NOETHER'S THEOREM

The usual statement of Noether's theorem is that for every continuous symmetry of the equations of motion (determined by the Lagrangian or Hamiltonian) there exists a conserved quantity. Although this is easy to prove, the meaning of this theorem is more readily evident

in the Hamiltonian than in the Lagrangian formulation. A symmetry of the equations of motion or time evolution is a transformation s such that in any experiment if s is applied to the initial state of the physical objects (fields, particles, etc.) participating in the experiment then the final state of the transformed experiment must be the same as s applied to the final state of the original experiment. For example, spatial translational symmetry implies that if the apparatus is translated to a new spatial location then the same experiment should give the same result.

Suppose U is the time-evolution operator, and ψ_i, ψ_f are the initial and final states, i.e. $\psi_f = U\psi_i$. Then the above definition of s being a symmetry of the time evolution is

$$s\psi_f = Us\psi_i$$

for every initial state ψ_i . This is equivalent to the commutator

$$[U, s] = 0. \tag{1}$$

But (1) states also that s is conserved during the time-evolution. Therefore, the statements that s is a symmetry and that s is conserved are the *same* statement (1), and there is nothing to prove!

Now suppose that there is a continuous symmetry generated by Q . Then (1) is satisfied with $s = \exp(iQq)$ for all q , and therefore

$$[U, Q] = 0. \tag{2}$$

Hence, Q is conserved, which is Noether's theorem. Furthermore, if the Hamiltonian H is independent of time t , as it is for an isolated system, then $U = \exp(\frac{-i}{\hbar}Ht)$. If (2) is valid for all t , then

$$[H, Q] = 0. \tag{3}$$

The above results may be extended to classical physics by turning the above commutators into Poisson brackets in classical phase space. These classical results may be regarded as

the classical limit of the quantum results by recognizing that the symplectic structure that gives the Poisson brackets are the classical limit [6] of a symplectic structure in quantum theory [7] that gives the commutators.

But the conservation of s that follows from (1) is more general than the usual form of Noether's theorem. There are at least two situations in which (1) is valid but there are no corresponding (2) or (3). First, as is well known, in both classical and quantum physics, s may be a discrete symmetry instead of a continuous symmetry. For example, s may be parity, which is a symmetry and therefore conserved for all interactions except the weak interaction, as far as we know. Another example is that (1) is satisfied for $s = \exp(iQq_k)$ for a discrete set of values q_k only. Second, in quantum physics the mean value of $s = \exp(iQq)$ has more information than the mean value of all the moments of Q , namely Q^n where n is any positive integer [3]. This is unlike in classical physics where the mean value of a transformation generated by Q may be obtained using the mean values of all the Q^n . Both these situations will be considered in section 5.

Since in (1) U and s occur symmetrically, it follows that the converse of the generalized Noether's theorem is also true: *A transformation s that is conserved must be a symmetry of the equations of motion.* The usual view is that U is more fundamental than s because U is determined by the dynamical laws which are regarded as primary, whereas the symmetries such as s are obtained secondarily as the symmetries of these laws. But the concise form (1) of the connection between the dynamical laws and symmetries, in which U and s are on an equivalent footing, suggest that we may equally turn the usual view around and regard the transformations $\{s\}$ as primary and U as derived from them to satisfy (1) so that $\{s\}$ are the symmetries [8]. The possibility of regarding symmetries as fundamental relations between quantum states by associating them with a quantum geometry will be explored in the next section.

III. QUANTUM GEOMETRY

The concept of space originates from our common experience of translating objects and from the possible states they can occupy. If we translate a cup, for example, in various possible ways, classically we may say that the different configurations or states of the cup are “immersed” in “space”. This space is *universal* in the sense that it is regarded as independent of the objects which are “contained” in it.

But quantum mechanically it is not clear what is meant by the cup being “immersed” in space. The cup consists of electrons, protons and neutrons (or the quarks and gluons which make up the protons and neutrons), and the states of these particles belong to the corresponding Hilbert spaces which are different from the physical space or the phase space of classical physics. The translation of a cup therefore needs to be represented by the corresponding translation operators that act on these Hilbert spaces. The fact that all the particles constituting the cup move together in some approximate sense suggests the introduction of universal translation group elements that are represented by operators that act on each Hilbert space. It is this *universality* of the translation group that gives us the concept of “space” that is independent of the particular system that partakes in it.

Also, it is well known that we cannot operationally determine the metric in space- time, or even the points of space-time, using quantum probes [9] because of the uncertainty principle. If one tries to obtain the space-time geometry using a clock and radar light signals, which is possible in classical physics [10], the uncertainty in the measurement of time in quantum physics is $\sim \hbar/\Delta E$, where ΔE is the uncertainty in the energy of the clock. If we try to decrease this uncertainty by increasing ΔE , then ΔE causes a corresponding uncertainty in the geometry of space- time. The total uncertainty in the measurement of space-time distances is then $\hbar c/\Delta E + 2G\Delta E/c^4$. The minimum value of this uncertainty as ΔE is varied is $\sim \text{Planck-length} = \sqrt{G\hbar/c^3}$. Hence, space- time geometry is only approximately valid in quantum theory, with an uncertainty of the order of Planck length.

However, a geometry for quantum theory may be defined by relations determined by a

universal group S , which generalizes the above translation group. This is universal in the sense that the same S has a representation in each Hilbert space. But S may have subgroups which may have trivial representations in some Hilbert spaces and not in others. An object may be displaced by any $s \in S$, which means that s acts on each of the Hilbert spaces of the particles or fields constituting that object through the corresponding representation of S . Each ψ in each of these Hilbert spaces is mapped to a corresponding ψ_s by this action of s , and the relation between ψ and ψ_s that is determined by s is regarded as independent of the Hilbert space and is therefore universal. In the example of a cup considered above, s is an element of the translation group, ψ and ψ_s are the states of each particle constituting the cup before and after the translation, and the relation between each such pair is universal in the sense that the entire cup has undergone this translation, or any other object that could take the place of the cup. This quantum geometry cannot be subject to the above criticisms of the space-time geometry because the action of S on each Hilbert space is not subject to any uncertainty.

It is reasonable to require that this geometry is preserved in time in the absence of interactions. Then each $s \in S$ is conserved and the converse of Noether's theorem stated in the last paragraph of the previous section implies that s is also a symmetry of the time evolution. Since the evolution equations are now determined by the standard model, S may be the symmetry group of the present day standard model, namely $P \times U(1) \times SU(2) \times SU(3)$, where P is the Poincare group. But if the standard model is superseded by new physics that has a different symmetry group then S should be this new group and the above statements would all be unaffected.

As an illustration of the geometrical relations proposed here, consider the experimentally known quantization of electric charge: all known charges are integral multiple of the fundamental charge e_0 . An aspect of this is that the magnitudes of the charges of the electron and the proton are experimentally known to be equal to an amazing precision. To obtain charge quantization, take s above to be an arbitrary element of the electromagnetic $U(1)$ group, which is a subgroup of S . This universal $U(1)$ group is a circle parametrized by Λ , say, that

varies from 0 to Λ_0 so that 0 and Λ_0 represent the same point on this group, chosen to be the identity. Since $U(1)$ is abelian, it has only one-dimensional representations. Hence the action of $s(\Lambda)$ on an arbitrary state gives

$$\psi_s = \exp(iQ\Lambda)\psi, \quad (4)$$

where Q corresponds to the particular representation of $U(1)$ in the Hilbert space in which ψ belongs to. But since $s(\Lambda_0) = s(0)$, which is due to the compactness of the $U(1)$ group, $\exp(iQ\Lambda_0) = 1$ for all representations [11]. Hence, $Q\Lambda_0 = 2\pi n$ or

$$Q = n \frac{2\pi}{\Lambda_0}, \quad (5)$$

where n is an integer.

To interpret Q , consider the physical implementation of the transformation s . This may be done by sending each of the particles through the same electromagnetic field with 4-vector potential A_μ in a particular gauge so that the effect of the electromagnetic field alone on the particle is given by

$$\psi_s = \exp(-i \frac{q}{\hbar c} \int A_\mu(x) dx^\mu) \psi, \quad (6)$$

which is a $U(1)$ transformation. Indeed, the statement that the electromagnetic field is a $U(1)$ gauge field may be taken to mean that it is physically possible to implement a $U(1)$ gauge transformation using the electromagnetic field in this way. Then q has the interpretation of the electric charge. Comparing (6) with (4), we may take Λ to be $\int A_\mu dx^\mu$ in which case $Q = \frac{q}{\hbar c}$. Hence, from (5),

$$q = ne_0 \quad (7)$$

where $e_0 = \frac{2\pi\hbar c}{\Lambda_0}$ is a universal constant that is determined experimentally to be $\frac{1}{3}e$, where e is the charge of the electron. The exact equality of the magnitudes of the charges of the electron and the proton may now be understood as due to them belonging to representations corresponding to $n = 3$ and $n = -3$, respectively.

The above argument also provides a reason for the introduction of Planck's constant which is purely geometrical. The exponent in (6) must be dimensionless because the expansion of the exponential has all powers of the exponent. Now, $\frac{q}{c} \int A_\mu dx^\mu$ is meaningful in classical physics. But to turn it into a dimensionless quantity, it is necessary to introduce a new scale, which is provided by \hbar . From the present point of view, this is needed in order to physically implement the $U(1)$ group elements that define relations between states which are part of the quantum geometry. Also, from (7), q is proportional to e , and A_μ is also proportional to e because the charges that generate A_μ via Maxwell's equations are proportional to e . Hence, the exponent in (6) is proportional to the fine-structure constant $\frac{e^2}{\hbar c}$. This argument may be extended to gauge fields in general, and dimensionless coupling constants are obtained for all of them.

The relation defined by (6) is not gauge invariant. Hence, it cannot be used to define an invariant geometrical 'distance'. Consider again the translation of a cup which may be performed by acting on all the quantum states of the particles constituting the cup by a universal group element $\exp(-\frac{i}{\hbar}\hat{p}\ell)$, where \hat{p} is a generator of translation. The action of this group element on a wave function is also not gauge invariant. But we may combine the transformations to define the gauge-covariant relation $\psi_f(x) = f(\ell)\psi(x)$, where

$$f(\ell) = \exp(\frac{i}{\hbar}\hat{p}_\mu\ell^\mu) \exp(-i\frac{q}{\hbar c} \int_x^{x+\ell} A_\mu(y)dy^\mu) \quad (8)$$

with ℓ^μ being a 4-vector, \hat{p}_μ the four generators of translation, and x and ℓ in the integral stand for x^ν and ℓ^ν , respectively, and the metric convention is $(+, -, -, -)$. The operator (8) is observable. For example, in the Josephson effect, where the current depends on the gauge invariant phase difference across the junction, if ℓ^μ is chosen to be the space-like vector across the junction then $f(\ell)$ is observable from the current [12]. We may generalize it to an arbitrary gauge field with vector potential A_μ^k for the gauge field integral, by defining the relation $\psi_g(x) = g(\gamma)\psi(x)$, where the gauge-covariant operator g is

$$g(\gamma) = \exp(\frac{i}{\hbar}\hat{p}_\mu\ell^\mu)P \exp(-ig_0 \int_\gamma A_\mu^k T_k dx^\mu), \quad (9)$$

where T_k generate the gauge group, P denotes path ordering, and γ is the geodesic joining x and $x + \ell$. Under a gauge transformation $u(x)$, $g(\gamma)$ transforms to

$$g'(\gamma) = \exp\left(\frac{i}{\hbar}\hat{p}_\mu\ell^\mu\right)u(x+\ell)P\exp\left(-ig\int_\gamma A_\mu^k T_k dx^\mu\right)u^\dagger(x) = u(x)g(\gamma)u^\dagger(x). \quad (10)$$

Also,

$$\begin{aligned} \langle \psi | g(\gamma) | \psi \rangle &= \langle \exp\left(-\frac{i}{\hbar}\hat{p}_\mu\ell^\mu\right)\psi | P\exp\left(-ig_0\int_\gamma A_\mu^k T_k\right) | \psi \rangle \\ &= \int d^3x \psi^\dagger(x+\ell)P\exp\left(-ig_0\int_\gamma A_\mu^k T_k\right)\psi(x) \end{aligned} \quad (11)$$

which is explicitly gauge invariant, because the integrand is gauge invariant. It may be observable, in principle, by the Josephson effect for a non Abelian gauge theory proposed in [12].

In (9) it is assumed that $\hat{p}_0 c = H$ commutes with $p_i = i\hbar\frac{\partial}{\partial x^i}$ and \hat{p}_0 at different times commute with each other. But if the system is interacting with the external fields this is not true in general. Then the spatial and time-translations will depend on which order they are done. Therefore, $\exp(\frac{i}{\hbar}\hat{p}_\mu\ell^\mu)$ in (9) will then need to be replaced by the path ordered operator $P\exp(\frac{i}{\hbar}\int_\gamma \hat{p}_\mu dy^\mu)$. Also, γ may be generalized to any piecewise smooth curve connecting x and $x + \ell$. The latter operator depends on the path γ even for fixed endpoints.

The relation between ψ_g and ψ provided by g is gauge and Lorentz covariant and is observable in principle. And it may be reasonable to take it as a quantum distance between the two states.

IV. CLASSICAL LIMIT

To take the classical limit of this geometry, note that classical space-time is constructed with measuring instruments consisting of particles that have approximate position and momentum. It is therefore reasonable to represent them by Gaussian wave packets which have minimum uncertainty. For a particle with mean position at the origin and mean momentum zero, the normalized wave function of such a state up to an arbitrary phase factor is

$$\psi_{\mathbf{0}}(\mathbf{x}) = (2\pi\Delta x^2)^{-1/4} \exp\left(-\frac{\mathbf{x}^2}{4\Delta x^2}\right) \quad (12)$$

where Δx is the uncertainty in position. This may be a state of a molecule in the cup, mentioned in section 3, in a harmonic oscillator potential in which case it would not spread. As the cup is displaced, the above wave function becomes

$$\psi_{\ell}(\mathbf{x}) \equiv \exp\left(-\frac{i}{\hbar}\mathbf{p} \cdot \ell\right)\psi_{\mathbf{0}}(\mathbf{x}) = (2\pi\Delta x^2)^{-1/4} \exp\left(-\frac{(\mathbf{x} - \ell)^2}{4\Delta x^2}\right) \quad (13)$$

up to a phase factor.

In (9), the second factor may be made the identity for a given γ by choosing a gauge in which $A_{\mu}^k \ell^{\mu} = 0$. Therefore, $\exp(-i\mathbf{p} \cdot \ell)$ may be regarded as a special case of (9), and hence as defining a quantum distance between the two states $\psi_{\mathbf{0}}$ and ψ_{ℓ} . We may therefore expect a metric to be defined on the translation group to which this operator belongs to and use that to define a metric in space. But this group, being Abelian, has no natural metric on it. There are two ways, however, that a metric may be defined on it. One is to use the Casimir operator of the Poincare group, $\eta_{ab}P^aP^b$, to define a metric on it, which locally may be associated with the space-time metric [1]. The other method, which will be used here, is to utilize the overlap of the two wave functions to obtain a measure of the displacement between them, which would then give an equivalent metric in the translation group. This is possible if the space of wavefunctions on which this group acts has an inner product, which would give a measure of the overlap and therefore how far a state has been translated.

Such a measure is given by the Fubini-Study metric in the quantum state space, or the set of rays, of every Hilbert space. This is the unique metric, up to multiplication by an overall constant, that is invariant under unitary (and anti-unitary) transformations. This may therefore be written in the form [13,14,6]

$$dS^2 = 4(1 - |\langle \psi | \psi' \rangle|^2) \quad (14)$$

where dS is the infinitesimal distance between two neighboring states (rays) represented by normalized state vectors ψ and ψ' . Clearly, dS is zero when the states are the same, and it

increases when the overlap between the states decreases. It is also invariant under unitary transformations, and must therefore be the Fubini- Study metric. The factor 4 in (14) is just a convention which ensures that this metric in the state space of the Hilbert space spanned by ψ and ψ' is the metric on a sphere of unit radius.

Now substitute $\psi_\ell(\mathbf{x})$ and $\psi_{\ell+d\ell}(\mathbf{x})$ as ψ and ψ' in (14). Then,

$$dS^2 = \frac{d\ell^2}{\Delta x^2} \quad (15)$$

neglecting higher order terms in $d\ell$ because it is infinitesimal. Hence $d\ell^2$, which is the same for all Hilbert spaces, may be used as a metric on the 3 dimensional translational group that is parametrized by the components of the vector ℓ . Locally, this metric may be regarded as a metric in the physical space of classical physics.

This result may be generalized to an arbitrary state ψ as follows. Require that $\langle \psi | \psi' \rangle$ to be close to the identity, where

$$\psi'(\mathbf{x}) \equiv \exp\left(-\frac{i}{\hbar} \mathbf{p} \cdot d\ell\right) \psi(\mathbf{x}). \quad (16)$$

Therefore expand

$$\langle \psi | \psi' \rangle \simeq \langle \psi | \left(1 - \frac{i}{\hbar} p d\ell - \frac{1}{2\hbar^2} p^2 d\ell^2\right) \psi \rangle = 1 - \frac{i}{\hbar} \langle \psi | p | \psi \rangle d\ell - \frac{1}{2\hbar^2} \langle \psi | p^2 | \psi \rangle d\ell^2 \quad (17)$$

where p is the momentum component in the direction of $d\ell$. Substituting this in (14),

$$dS^2 = \frac{4\Delta p^2}{\hbar^2} d\ell^2 \quad (18)$$

where Δp is the uncertainty in p : $\Delta p^2 = \langle \psi | p^2 | \psi \rangle - \langle \psi | p | \psi \rangle^2$. For the Gaussian wave packet, $\Delta p \Delta x = \hbar/2$ and therefore (18) gives (15) in this case.

Time is measured by a clock. Since the clock must have moving parts, the uncertainty ΔE of its Hamiltonian H must be non zero. Neglecting any external interaction of the clock, H is a constant. The infinitesimal time evolution of the the clock is given by

$$|\psi(t + dt)\rangle = \exp\left(-\frac{i}{\hbar} H dt\right) |\psi(t)\rangle \quad (19)$$

The Fubini-Study distance dS along the evolution curve in the quantum state space corresponding to $|\psi(t)\rangle$ is obtained analogous to the derivation of (18) to be [14,6]

$$dS^2 = \frac{4\Delta E^2}{\hbar^2} dt^2 \quad (20)$$

A quantum clock directly measures the Fubini-Study distance S and the time t is then inferred from S using (20). The appearance of the same t in (20) in all Hilbert spaces is due to the universality of the action of the time translation $\exp(\frac{i}{\hbar}Hdt)$ in every Hilbert space. This is analogous to the universality of the spatial displacements which was used earlier to obtain the spatial metric.

The spatial translation (16) and the active time translation corresponding to (19) may be written covariantly as

$$\psi_\ell = \exp(\frac{i}{\hbar}p_\mu d\ell^\mu)\psi_o \quad (21)$$

where $d\ell^\mu = (cdt, d\ell)$. This is the infinitesimal version of the transformation (9) in a special gauge, as mentioned earlier. In an arbitrary gauge, the same space and time metrics are obtained by replacing (21) with (9) in the above treatment. In relativistic quantum mechanics, owing to the transformation property of $p_\mu = (H/c, p_i)$, these metrics give a space-time metric which is Lorentzian. In the presence of gravity, this metric may be obtained locally by doing the above space-time translations in a freely falling Einstein elevator.

V. INTERACTIONS AND THE NON LOCALITY OF QUANTUM THEORY

If the quantum geometry is determined by relations between states that are group elements, and if these group elements, which are our observables, are made dynamical the way Einstein made space-time distances dynamical in order to obtain gravity, then this would give both gravity and gauge fields [8]. Also, quantum mechanics has an inherent non locality, which may also be understood in terms of these group elements. The combination of these two statements imply that gauge fields and gravity should affect quantum states in a non local manner as in the AB effect [4], as will be discussed later.

First consider the non locality of quantum theory. This may be illustrated by the following example studied by Aharonov, et al [3]. Electrons with initial momentum in the x -direction go through an infinite diffraction grating in the yz -plane of a Cartesian coordinate system, with the length of the slits along the z -direction. Then the grating destroys continuous translational symmetry for the electrons in the y -direction. However, if the distance between successive slits in the y direction is ℓ then $s = \exp(-i\frac{p\ell}{\hbar})$ satisfies (1), where p is the momentum operator for electrons in the y -direction which generates translations in the y -direction. Hence, it follows from the generalized Noether's theorem in section 2 that $\exp(-i\frac{p\ell}{\hbar})$ is conserved although p is *not* conserved. Indeed, it is well known that the interference fringes on a screen that is parallel to and far away from the yz - plane is given by $\ell \sin \theta_n = n\lambda$, where λ is the wave length and n is an integer. Therefore, the possible values of the momentum for an electron in the y - direction after the interaction are $p_n = \frac{h}{\lambda} \sin \theta_n = n\frac{h}{\ell}$, i.e. $\exp(-i\frac{p_n\ell}{\hbar}) = 1$. Hence, $\exp(-i\frac{p\ell}{\hbar})$ is conserved during the passage of electrons through the grating.

The operator $m \equiv \exp(i\frac{p\ell}{\hbar})$ is equivalent to the modular momentum $p(mod\frac{h}{\ell})$ introduced by Aharonov,et al [3]. But here I shall treat m as an element of a universal group that is used to define a quantum geometry as in section 3. m may be obtained from experiments by measuring the Hermitian observables

$$s_R \equiv \frac{1}{2}[\exp(i\frac{p\ell}{\hbar}) + \exp(-i\frac{p\ell}{\hbar})], s_I \equiv \frac{1}{2i}[\exp(i\frac{p\ell}{\hbar}) - \exp(-i\frac{p\ell}{\hbar})]. \quad (22)$$

Therefore, the unitary operator m may also be regarded as an observable. It is important to note that this is a *non-local* observable, unlike p .

This non locality in quantum mechanics may be illustrated also in the simple interference experiment of two coherent wave packets. Suppose the two wave packets are moving in the x -direction and are the same at time t except that their centers are separated by a displacement ℓ in the y -direction and there is a phase difference α between the wave packets, where α is a constant. The wave is then a superposition of two wave packets:

$$\psi(x, y, z, t) = \frac{1}{\sqrt{2}}\{\phi(x, y - \ell, z, t) + e^{i\alpha}\phi(x, y, z, t)\} \quad (23)$$

Now no local experiments performed on the two wave packets at the two slits could determine the phase factor $e^{i\alpha}$. For example, the expectation values of the local variables p^n , where n is any positive integer, give no information about $e^{i\alpha}$ [3]. This is easily verified by writing $p^n = (-i\hbar \frac{\partial}{\partial y})^n$ in the coordinate representation. But

$$\langle \psi | \exp(-i \frac{p\ell}{\hbar}) | \psi \rangle = \frac{e^{i\alpha}}{2} \quad (24)$$

This means that the momentum distribution at time t does depend on the phase factor $e^{i\alpha}$, i.e. if p is measured then the probability distribution for obtaining the individual eigenvalues of p is changed by this phase factor. And this may be experimentally verified by letting the wave packets interfere and observing the shift in interference fringes. Hence, $\langle \psi | \exp(-i \frac{p\ell}{\hbar}) | \psi \rangle$ contains more information than the expectation values $\langle \psi | p^n | \psi \rangle$ of any of the moments of momentum p^n . This is basically due to the linear structure of the Hilbert space, which physically corresponds to the principle of superposition.

This fundamental non locality of quantum mechanics translates into a non locality of the effect of all the fundamental interactions on the wave function. This has been shown for the AB effect due to a magnetic field by Aharonov et al [3]. It may be illustrated in the above described interference of two wave packets as follows. Suppose the above two wave packets A and B are those of an electron and they pass on the two sides of a solenoid containing a magnetic flux Φ . The gauge may be chosen so that the vector potential is non zero only along a thin strip bounded by two planes indicated by the dotted lines in figure 1. Then when the wave packets have passed the solenoid there is a phase difference between the wave packets given by $\alpha = \frac{e}{\hbar c} \Phi$. Therefore, the expectation value of the modular momentum g , which was 1 before the wave packets passed the solenoid is now given by (24). It has also pointed out by Aharonov [15] that in the above statement m may be replaced by the gauge invariant modular kinetic momentum f given by (8). This is because before and after the wave packets pass the solenoid the vector potential is zero and therefore f is the same as m . Since f is gauge covariant, the same result is obtained in every gauge.

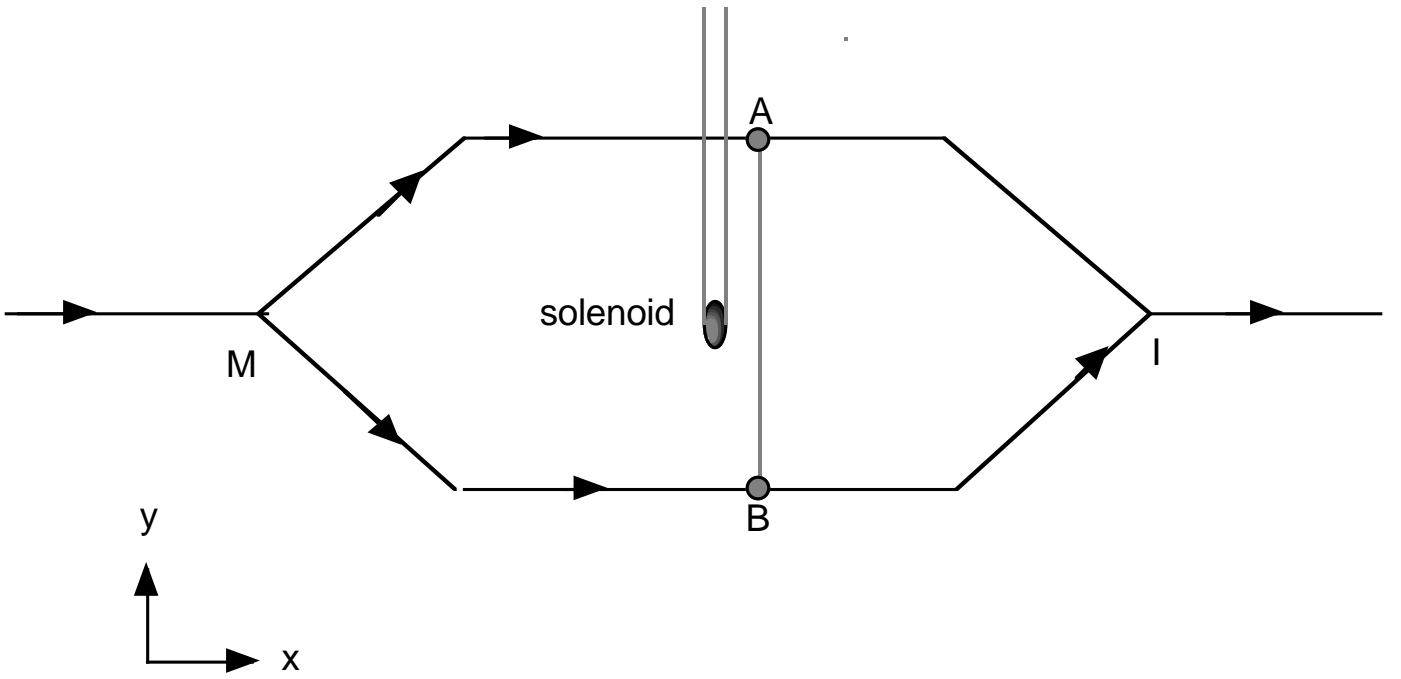


Figure 1. Vector Aharonov-Bohm effect in which a wave packet of an electron is split coherently into two wave packets at the beam splitter M which are then made to interfere at I . When the imaginary line joining the wave packets A and B sweeps across the magnetic flux in the solenoid (shaded region) the modular momentum or the modular kinetic momentum associated with this line changes, as pointed out by Aharonov [15].

Consider now the scalar Aharonov-Bohm effect. A wave packet traveling in the x -direction is partially transmitted and reflected by a beam splitter M (figure 2a). The resulting two wave packets which travel in opposite directions are reflected by two mirrors M_1 and M_2 situated along the x -axis and they interfere subsequently. Meanwhile a pair of oppositely charged capacitor plates C is separated and closed so that there is a non zero electric field in the region enclosed by the world- lines of the centers of the wave packets in the xt -plane. The same experiment is viewed in the rest frame of the reflected wave packet in figure 2b.

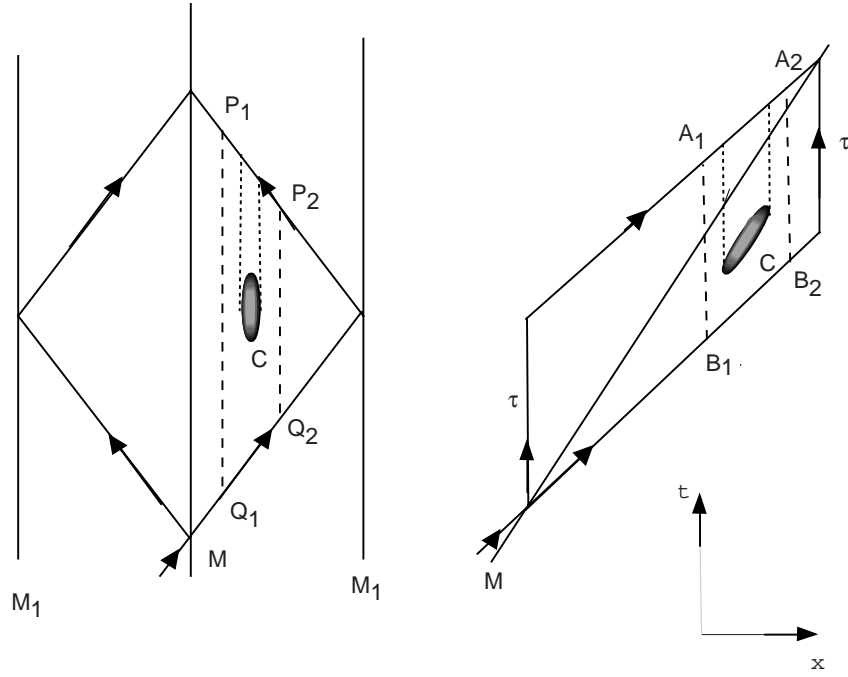


Figure 2. A scalar Aharonov-Bohm effect shown schematically in the tx -plane. (a) The modular energy and modular kinetic energy associated with the imaginary lines P_1Q_1 and P_2Q_2 are different, partly because of the scalar AB phase shift due to the region of non zero electric field in the capacitor that is open and shut (shaded region). (b) The same experiment viewed in the rest frame of the wave packet that is reflected at the beam splitter M . When the imaginary line AB joining the wave packets sweeps across the region of non zero electric field in the capacitor the modular energy and modular kinetic energy associated with this line changes due to the AB phase shift.

We may choose a gauge in which the vector potential is non zero only along a strip between the dotted lines parallel to the time-axis in figure 2a or 2b. Then the wave packet A at time $t + \tau$ develops a phase shift β with respect to B at time t as the imaginary line AB crosses the space-time region containing the electric field, where

$$\beta = \frac{e}{\hbar} \int_C F_{0x} dx dt \quad (25)$$

and C is the region in which the electric field $E = F_{0x}$ is non zero. This is a non local effect which may be understood using $U = T \exp(\frac{i}{\hbar} \int_0^\tau H dt)$, where T denotes time ordering and

τ is the time interval between the events A_1 and B_1 . U^\dagger corresponds to the modular energy of Aharonov, et al [3]. Suppose the wave function of the electron is

$$\psi = \frac{1}{\sqrt{2}}(\psi_A + \psi_B)$$

where ψ_A and ψ_B are the wave functions of the above two localized wave packets. Then

$$\langle \psi(t) | U | \psi(t) \rangle = \frac{1}{2} \langle \psi_A(t) | U | \psi_B(t) \rangle = \frac{1}{2} \langle \psi_A(t + \tau) | \psi_B(t) \rangle \quad (26)$$

changes by $e^{i\beta}$ due to the electric field E in the region C as the imaginary line AB sweeps across the small region C where E is non zero.

In the last statement and in (26) we may replace U by the gauge covariant modular kinetic energy operator $V = T \exp(\frac{i}{\hbar} \int H dt) \exp(-\frac{i}{\hbar} \int e A_0 dt)$, which is a special case of (9). This is because A_0 is zero along $A_1 B_1$ and $\overline{A_1 B_1}$, which are outside the strip that contains the non zero A_μ . However, $\langle \psi(t) | V | \psi(t) \rangle$ is gauge invariant and therefore its change mentioned above may be obtained in any gauge. But since there are no forces acting on the electron, there is no change of its kinetic energy $H - e A_0$ or any of its moments. Hence, $\langle \psi(t) | V | \psi(t) \rangle$ contains non local aspects which cannot be obtained by measurements of the kinetic energy operator or any of its moments. Thus the scalar AB effect may be viewed as a quantum effect which is *non local in time*.

The above results are easily generalized to the Aharonov-Bohm effects due to non Abelian gauge fields by replacing the electromagnetic fluxes by Yang-Mills fluxes, and using the expectation values (11) of the generalized modular kinetic energy-momentum (9).

The group element (9) belongs to the group $T_4 \times G$, where T_4 is the translation group and G is the gauge group. But it has an asymmetry in that the part of (9) that belongs to G is dynamical, whereas the part that belongs to T_4 is fixed. Since I proposed that the fundamental interactions should correspond to the universal group element (9) being dynamical, consistency requires that the part of (9) that belongs to T_4 should be dynamical as well, i.e. ℓ^μ should be made dynamical. But the classical space-time geometry was constructed in section 4 using the latter group elements. It follows therefore that making ℓ^μ

dynamical would make the space-time metric dynamical and not fixed as it is in Minkowski space-time. Therefore, the interaction that corresponds to making the T_4 group elements dynamical gives, in the classical limit, the well known geometrical description of gravity in classical general relativity. A generalization of it is obtained by replacing T_4 with T_n , where n is any positive integer. Thus the present approach requires the existence of gravity.

I now give a simple illustration of the above unified way of treating gravity and gauge fields by considering the gravitational analog of the above vector AB effect. The geometry surrounding a cosmic string in the two dimensional section normal to the axis of the string at a given time is that of a cone whose center is at the axis, which is seen by solving the classical gravitational field equations [16]. The space-time geometry of a non- rotating cosmic string is obtained by simply adding to this plane the extra dimension in the direction of the axis and the time dimension so that the curvature outside the string is zero everywhere (figure 3). It is known that this geometry is similar to the electromagnetic field around a solenoid because the curvature is zero outside the string and yet there is a non trivial holonomy around it.

Consider now two wave packets separated by a distance ℓ and whose centers move along initially parallel lines such that the geodesic line AB meets the conical singularity S at its midpoint. But there is another geodesic that connects the same pair of points A and B of length $\ell \cos(\frac{\theta}{2})$, shown by the line $A'B$ in figure 3, where A' is identified with A . (Actually, there are two geodesics connecting A and B when the angle $\hat{A}SB$ on the left side exceeds $\pi - \theta$ but is less than π . The corresponding angles $\hat{A}SB$ on the right side are π and $\pi - \theta$, respectively. The singularity may be replaced by a small cap that merges with the rest of the cone smoothly. Then there would be a third geodesic joining A and B though this cap.) Just after AB crosses the conical singularity, there is only one geodesic joining A and B , whose length is $\ell \cos(\theta/2)$. This is somewhat analogous to the change in modular momentum kinetic momentum when AB crosses the solenoid in fig. 1, because $\exp(-i\mathbf{p} \cdot \ell)$ now translates locally along parallel geodesic line segments of length ℓ , and this length has gotten shorter after the crossing.

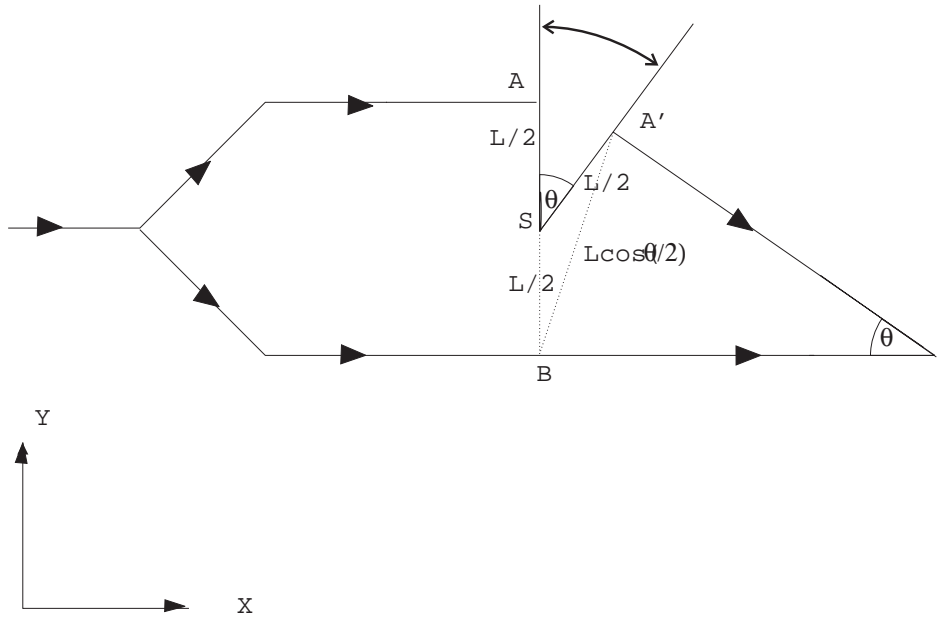


Figure 3. A gravitational analog of the experiment in fig. 1. The conical geometry around a cosmic string that is normal to the plane through S is represented by cutting off the wedge ASA' from flat space and identifying the planes along which it is cut. The wave packets moving at A (same as A') and B are focused by this geometry to interfere at I . Just before AB crosses S , there are two geodesics connecting A and B of lengths ℓ and $\ell \cos(\theta/2)$ in the curvature free region. But just after the crossing there is a unique geodesic $A'B$ of length $\ell \cos(\theta/2)$ joining A and B .

In figure 4, this result is generalized to the case of S not being the midpoint of the geodesic AB , and it is also seen to be ‘gauge’ independent in the sense of being invariant under the rotation of the ‘wedge’ mentioned above. There is then a phase shift $\Delta\phi$ due to the difference in path lengths traveled by the wave packets given by

$$\Delta\phi = \frac{p_0}{\hbar}(d_1 - d_2) \simeq \frac{p_0}{2\hbar}(\ell_2 - \ell_1)\theta = 4\pi G\mu \frac{p_0}{\hbar}(\ell_2 - \ell_1) \quad (27)$$

for small θ , where d_1 and d_2 are the path lengths AI and BI , μ is the mass per unit length of the cosmic string, and p_0 is the initial momentum of the particle. If the particle carries spin, then there is also a phase shift due to the coupling of spin to the curvature. This and other phase shifts for interference of two wave packets around a cosmic string are

studied elsewhere and may be understood as being due to the affine holonomy around the string [17,16].

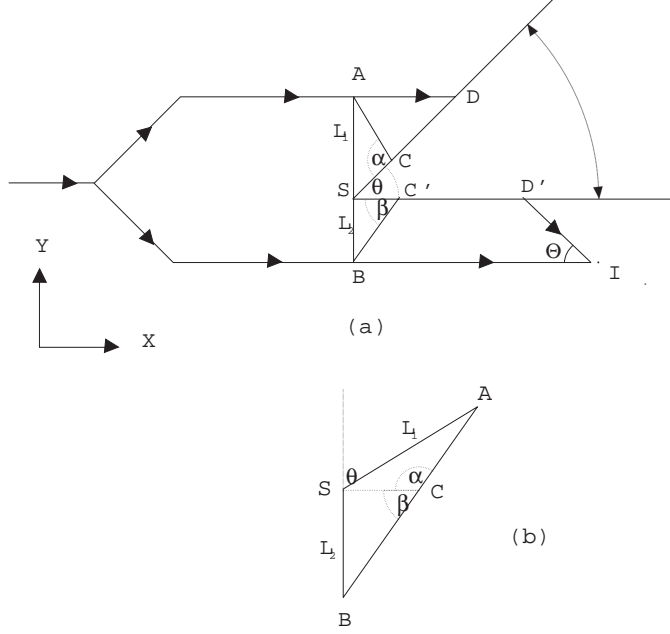


Figure 4. A generalization of the gedanken experiment shown schematically in fig. 3 to unequal lengths $AS = \ell_1$ and $BS = \ell_2$. The wedge DSD' has an arbitrary orientation. But since SC and SC' are identified, and $\alpha + \beta = \pi$ in order for ACB to be a geodesic, the distances along the geodesics ASB and ACB are respectively $\ell_1 + \ell_2$ and $(\ell_1^2 + \ell_2^2 + 2\ell_1\ell_2 \cos \theta)^{1/2}$, as shown in fig. (b), independently of the orientation of the wedge.

VI. DISCUSSION AND CONCLUSION

The change in geodesic distances between the wave packets due to the string, in section 5, is not surprising because gravity changes distances, according to general relativity, and the cosmic string is a purely general relativistic object without a Newtonian analog. What may be more interesting is the similar change of the ‘quantum distances’ studied earlier due to the electromagnetic field in the usual AB effect and its generalization to non Abelian gauge fields. Both these effects may be treated in a somewhat analogous manner if the modular kinetic energy and modular kinetic momentum, regarded here as universal group elements, may be interpreted as ‘distances’ in a quantum geometry as proposed earlier. The fact

that space-time distances may be obtained from them approximately, as shown in section 4, reinforces this view.

The treatment of (9) as an observable, which implies the above mentioned non local effects in quantum theory, perhaps removes the mystery of why although the interactions are local as they occur in the Hamiltonian or Lagrangian, nevertheless there are non- local effects such as the Aharonov-Bohm effect and its generalizations to non abelian gauge fields and gravitation [18].

A criticism that may be made against regarding the universal group elements (9) as quantum distances is that their action on wave functions seems to require that γ be interpreted as a congruence of parallel space-time curves, whereas it was argued in section 3 that space-time geometry is not appropriate for quantum theory. However, as discussed in that section, this limitation becomes critical only at the Planck scales. But at the Planck scale these group elements need not be associated with curves in space-time. They may be defined simply as operators acting on quantum states defining the quantum geometry and representing the fundamental interactions. Thus it is possible to have a quantum geometry even at scales in which the space-time geometry breaks down and this would also give a quantum description of all the interactions.

ACKNOWLEDGMENTS

I thank Yakir Aharonov and Ralph Howard for useful discussions.

REFERENCES

- [1] J. Anandan, Foundations of Physics **10**, 601-629 (1980),
- [2] J. Anandan, Il Nuovo Cimento **53A.**, 221 (1979).
- [3] Y. Aharonov, H. Pendleton and A. Peterson, Int. J. Theoretical Phys. **2**, 213 (1969).
- [4] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).
- [5] D. Wisnievesky and Y. Aharonov, Ann. Phys. **45**, 479 (1967).
- [6] J. Anandan, Phys. Lett. A 147, 3 (1990); J. Anandan, Foundations of Physics, **21**, 1265 (1991).
- [7] T. W. B. Kibble, Communications in Mathematical Physics, **65**, 189 (1979).
- [8] J. Anandan, Foundations of Physics, vol. 29, no. 11, 1647- 1672 (1999). quant-ph/9808045 .
- [9] Eugene Wigner, *Symmetries and Reflections* (Indiana U. P., Bloomington/London, 1967)
- [10] J. L. Synge, *Relativity: The General Theory* (North Holland, Amsterdam, 1960), Chapter III.
- [11] C.N. Yang, Phys. Rev. D, **1**, 2360 (1970).
- [12] J. Anandan, Phys. Rev. D, **33**, 2280-2287 (1986).
- [13] J.P. Provost and G. Vallee, Commun. Math. Phys. **76**, 289 (1980).
- [14] J. Anandan and Y. Aharonov, Physical Review Letters **65**, 1697-1700 (1990),
- [15] Y. Aharonov , private communication.
- [16] J. Anandan, Phys. Rev. D, **53**, 779 (1996), gr-qc/9507049, and references therein.
- [17] J. Anandan, Physics Letters A, **195**, 284 (1994).

- [18] This reconciles the views of Y. Aharonov and C. N. Yang, who regarded the Aharonov-Bohm effect as being non local and local, respectively, in Proceedings of the International Symposium on the Foundations of Quantum Mechanics, Tokyo, August 1983, edited by S. Kamefuchi et al. (Physical Society of Japan, Tokyo, 1984) 65-73.